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The Solution of Some Non-Linear Integral Equations with Cauchy Kernels

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1. Introduction

In the theory of radiative transfer there are several problems which can be solved by finding the solution H(u) of the non-linear integral equation

$$(1.1) \qquad \frac{1}{H(x)} = 1 - x\int_0^1 \frac{g(u)H(u)du}{u+x}$$

which is called Chandrasekhar's equation. The books by Chandrasekhar [1] and Kourganoff [2] contain discussions of this important equation; and these books also present the contributions of various mathematicians who have shown that (1.1) can be solved explicitly by using function theory techniques based on analytic continuation. Recently, (1961) C. Fox [3] has shown that (1.1) can be converted into the linear equation

$$(1.2) \qquad H(x)G(x) = 1 + x\int_0^1 \frac{g(u)H(u)du}{u-x}$$

where G(x) is known and g(x) is prescribed. The equation (1.2) is a singular integral equation with a Cauchy kernel and it can be solved for H(u) by using an extension of Carleman's method as shown for example in Muskhelishvili's book [4].

Chandrasekhar's equation can be linearized by first writing it in the form

$$(1.3) \qquad xg(x) = \lambda\phi_1(x) + \phi_1(x)\int_0^1 \frac{\phi_1(u)du}{u+x}$$

and then integrating each side of this equation after it has been multiplied by the factor $1/(x+\xi)$. This gives

$$(1.4) \qquad \int_0^1 \frac{xg(x)du}{x+\xi} = \left[\lambda + \int_0^1 \frac{\phi_1(u)du}{u-\xi}\right] \int_0^1 \frac{\phi_1(x)dx}{x+\xi} - \int_0^1 \frac{\phi_1(u)}{u-\xi} \int_0^1 \frac{\phi_1(x)dx}{x+u}\, du$$

and after multiplying (1.4) by $\phi_1(\xi)$ and using (1.3) we find

$$(1.5) \qquad \phi_1(\xi)G_1(\xi) = \lambda + \int_0^1 \frac{\phi_1(x)dx}{x-\xi} .$$

This is essentially the procedure that was used by Fox to pass from (1.1) to (1.2). It suggests the possibility of solving

$$(1.6) \qquad \lambda\phi_1(x) + \phi_1(x) \int_0^1 \frac{\phi_1(u)du}{u-x} = f_1(x)$$

which is both singular and non-linear. Equation (1.6), in turn, suggests an investigation of the more general equation

$$\text{I} \qquad \lambda\phi(\zeta) + \phi(\zeta) \int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} = f(\zeta)$$

where ζ is an interior point of the simple smooth arc L which connects the points τ_0 and τ_1 in the complex τ-plane.

One of the purposes of this paper is to show in Section 2 that the equation I can be solved explicitly by using elementary function theory techniques. It turns out that the solution of I is in some ways simpler than the solution of (1.3). We will also be concerned with the solution of some other non-linear equations. In Section 3 we show how to solve

$$\text{II} \qquad \pi^2\phi^2(\zeta) - \left[\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta}\right]^2 = f(\zeta)$$

and Section 4 is devoted to the solution of

$$\text{III} \qquad \pi^2\phi^2(\zeta) + \left[\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta}\right]^2 = f(\zeta) .$$

In Section 5 we show that there is a connection between equations I, II, III and certain problems in potential theory.

The equations I, II, III may be of interest for at least two reasons. In the first place, they are of interest in themselves as Cauchy singular, non-linear integral equations which can be solved explicitly. In the second place, they present a formulation of certain non-linear boundary value problems. Equation III, for example, is intimately related to a problem in two-dimensional potential theory which has a number of physical applications. This is the problem of finding a potential function in a domain D when its normal derivative is prescribed on one part of the boundary C; and the magnitude of its gradient is given on the remaining part of C. In Section 5 we show how an explicit formula for the solution of this problem can be found.

The final Section 6 is concerned with a brief discussion of the non-linear system

$$\phi(\zeta)\int_L \frac{\psi(\tau)d\tau}{\tau-\zeta} = f(\zeta)$$

$$\psi(\zeta)\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} = g(\zeta)$$

and some other systems which can be linearized by the method developed in Section 2.

We state here the main conditions and assumptions upon which our analysis is based. If $\tau = \tau(t)$ is the equations of the simple smooth arc L directed from τ_0 to $\tau_1 \neq \tau_0$ let $L[\tau_0,\tau_1]$ denote the set of points $\tau = \tau(t)$, $t_0 \leq t \leq t_1$; and let $L(\tau_0,\tau_1)$ denote the set $L[\tau_0,\tau_1]$ minus the endpoints τ_0 and τ_1. We assume for simplicity that the unknown function $\phi(\tau)$ satisfies a uniform Hölder condition for any pair of points in $L(\tau_0,\tau_1)$ and thus guarantee the existence of the Cauchy principal values of the integrals which appear in I, II, III where ζ is in $L(\tau_0,\tau_1)$. We also assume that if $\phi(\tau)$ has a singularity at an endpoint α of $L[\tau_0,\tau_1]$ it is such that $\mathcal{L}_{\tau\to\alpha}(\tau-\alpha)\phi(\tau) = 0$. We will say that a function with these properties belongs to the class $\mathcal{H}_1$. If $\phi(\tau)$ belongs to the class $\mathcal{H}_1$ it can be shown that

$$\int_L \frac{\phi(\tau)d\tau}{\tau - \zeta}$$

satisfies a uniform Hölder condition for any pair of points ζ_1 and ζ_2 in $L(\tau_0,\tau_1)$; and

$$F(z) = \int_L \frac{\phi(\tau)d\tau}{\tau - z}$$

as a function of the complex variable z is such that $\mathcal{L}_{z\to\alpha}(z-\alpha)F(z) = 0$.

The properties of the prescribed function must match the properties implied by the representations on the left-hand side of I, II, III. That is, in accordance with our assumptions about

$\phi(\tau)$, $f(\zeta)$ must satisfy a uniform Hölder condition on $L(\tau_0,\tau_1)$; and although it may have a stronger singularity than $\phi(\tau)$ at α, $f(\zeta)$ must be such that $\mathcal{L}_{\zeta\to\alpha}(\zeta-\alpha)^2 f(\zeta) = 0$. We will say that under these conditions $f(\zeta)$ belongs to the class $\mathcal{H}_2$.

The transformation $2\tau = (\tau_1-\tau_0)\nu + (\tau_0+\tau_1)$ maps $L[\tau_0,\tau_1]$ into $L[-1,1]$ a simple smooth arc directed from $\tau = -1$ to $\tau = 1$. If we also use $2\zeta = (\tau_1-\tau_0)\omega + (\tau_0+\tau_1)$ we have

$$\int_{L[\tau_0,\tau_1]} \frac{\phi(\tau)d\tau}{\tau-\zeta} = \int_{L[-1,1]} \frac{\phi_0(\nu)d\nu}{\nu-\omega}$$

which shows that the transformation does not change the form of equations I, II, III. Thus there is no loss of generality if we assume, as we will hereafter, that L in I, II, III is $L[-1,1]$.

2. Equation I

We proceed to show how

$$(2.1) \qquad \lambda\phi(\zeta) + \phi(\zeta)\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} = f(\zeta)$$

can be solved by using the Hardy-Poincaré-Bertrand formula. This formula states that if $\psi(\tau,\zeta)$ is suitably restricted then

$$(2.2) \qquad \int_L \frac{1}{\zeta-\omega}\int_L \frac{\psi(\tau,\zeta)d\tau d\zeta}{\tau-\zeta} = -\pi^2\psi(\omega,\omega) + \int_L\int_L \frac{\psi(\tau,\zeta)d\zeta d\tau}{(\tau-\zeta)(\zeta-\omega)}$$

provided ω is in $L(-1,1)$. This says that an interchange of the order of integration on the left side of (2.2) leaves the residue $-\pi^2\psi(\omega,\omega)$. If $\phi(\tau)$ belongs to the class $\mathcal{H}_1$, defined in the

introduction, the formula (2.2) holds with $\psi(\tau,\zeta) = (1-\zeta^2)\phi(\zeta)\phi(\tau)$. The application of (2.2) to

$$\frac{1}{1-\omega^2}\int_L \frac{(1-\zeta^2)\phi(\zeta)}{\zeta-\omega}\int_L \frac{\phi(\tau)d\tau d\zeta}{\tau-\zeta}$$

shows, after a little manipulation, that

$$(2.3)\quad \frac{2}{1-\omega^2}\int_L \frac{(1-\zeta^2)\phi(\zeta)}{\zeta-\omega}\int_L \frac{\phi(\tau)d\tau d\zeta}{\tau-\zeta}$$

$$= -\pi^2\phi^2(\omega) + \left[\int_L \frac{\phi(\tau)d\tau}{\tau-\omega}\right]^2 + \frac{1}{1-\omega^2}\left[\int_L \phi(\tau)d\tau\right]^2 .$$

Now if a solution of (2.1) exists when $f(\zeta)$ belongs to $\mathcal{H}_2$, then (2.1) and (2.3) imply

$$(2.4)\quad \frac{2}{1-\omega^2}\int_L \frac{(1-\zeta^2)f(\zeta)d\zeta}{\zeta-\omega} = \frac{2\lambda}{1-\omega^2}\int_L \frac{(1-\zeta^2)\phi(\zeta)d\zeta}{\zeta-\omega} - \pi^2\phi^2(\omega)$$

$$+ \left[\int_L \frac{\phi(\tau)d\tau}{\tau-\omega}\right]^2 + \frac{1}{1-\omega^2}\left[\int_L \phi(\tau)d\tau\right]^2$$

or

$$(2.5)\quad \frac{2}{1-\omega^2}\int_L \frac{(1-\zeta^2)f(\zeta)d\zeta}{\zeta-\omega} = -\pi^2\phi^2(\omega) + 2\lambda\int_L \frac{\phi(\tau)d\tau}{\tau-\omega}$$

$$+ \left[\int_L \frac{\phi(\tau)d\tau}{\tau-\omega}\right]^2 + \frac{[k_o^2 - 2\lambda(k_1+\omega k_o)]}{1-\omega^2}$$

where

$$k_o = \int_L \phi(\tau)d\tau$$

$$k_1 = \int_L \tau\phi(\tau)d\tau .$$

The addition of

$$(2.6) \qquad \pm 2\pi i\ f(\omega) = \pm \left[2\lambda\pi i\ \phi(\omega) + 2\pi i\ \phi(\omega) \int_L \frac{\phi(\tau)d\tau}{\tau - \omega}\right]$$

to (2.5) gives

$$(2.7) \qquad \frac{2}{1-\omega^2} \int_L \frac{(1-\zeta^2)f(\zeta)d\zeta}{\zeta - \omega} \pm 2\pi i\ f(\omega)$$

$$= \left[\pm \pi i\ \phi(\omega) + \int_L \frac{\phi(\tau)d\tau}{\tau - \omega} + \lambda\right]^2 - \lambda^2 + \frac{[k_o^2 - 2\lambda(k_1 + \omega k_o)]}{1-\omega^2} .$$

Let us introduce the function

$$F_o(z) = (1-z^2)\left[\int_L \frac{\phi(\tau)d\tau}{\tau - z} + \lambda\right]^2$$

$$- \left[2\int_L \frac{(1-\tau^2)f(\tau)d\tau}{\tau - z} + 2\lambda(k_1 + zk_o) - k_o^2 + (1-z^2)\lambda^2\right] .$$

This function is analytic for z not on $L[-1,1]$ and it vanishes as $z \longrightarrow \infty$. As z approaches the point ω from the positive and negative sides of L, the limit values of $F_o(z)$ are

$$F_o^{\pm}(\omega) = (1-\omega^2)\left[\pm \pi i\ \phi(\omega) + \int_L \frac{\phi(\tau)d\tau}{\tau - \omega} + \lambda\right]^2$$

$$- \left[\pm 2\pi i(1-\omega^2)f(\omega) + 2\int_L \frac{(1-\tau^2)f(\tau)d\tau}{\tau - \omega}\right]$$

$$- [2\lambda(k_1 + \omega k_o) - k_o^2 + (1-\omega^2)\lambda^2]$$

but, as we can see from (2.7), these limit values vanish. Furthermore, it follows from the conditions we have imposed on ϕ and f, that the behavior of $F_o(z)$ in the neighborhood of an endpoint of L must be such that

$$\mathcal{L}_{z\to 1}(1-z)F_o(z) = 0 \tag{2.8}$$

$$\mathcal{L}_{z\to -1}(1+z)F_o(z) = 0 .$$

Let Γ be the boundary of a domain which contains L[-1,1]. If z is in the exterior of Γ then

$$F_o(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{F_o(\zeta)d\zeta}{\zeta - z} .$$

Since (2.8) holds, the path Γ can be collapsed into the path C, composed of the upper and lower banks of L, without changing the value of the Cauchy integral. At these banks, however, the limit values $F_o^{\pm}(\omega)$ vanish. We therefore conclude that $F_o(z)$ vanishes everywhere in the exterior of L[-1,1] and hence, if a solution of (2.1) exists, the constants k_o and k_1 must be such that the two-valued function

$$S(z) = \left[\frac{1}{1-z^2}\left\{2\int_L \frac{(1-\tau^2)f(\tau)d\tau}{\tau - z} + 2\lambda(k_1 + zk_o) - k_o^2 + (1-z^2)\lambda^2\right\}\right]^{\frac{1}{2}} \tag{2.9}$$

$$= \int_L \frac{\phi(\tau)d\tau}{\tau - z} + \lambda \tag{A}$$

is analytic for z not on L[-1,1]. We will refer to this condition as condition (A).

If for a prescribed $f(\zeta)$ constants k_o and k_1 exist such that $S(z)$ is analytic for z not on $L[-1,1]$, then, as we can see by taking the limit values of (2.9),

$$(2.10) \qquad \int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} + \pi i\, \phi(\zeta) + \lambda = S^+(\zeta)$$

$$(2.11) \qquad \int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} - \pi i\, \phi(\zeta) + \lambda = S^-(\zeta)$$

and, as subtraction of (2.11) from (2.10) shows, the solution of (2.1) is

$$\phi(\zeta) = \frac{1}{2\pi i}\,[S^+(\zeta) - S^-(\zeta)]$$

$$(2.12) \qquad = \frac{1}{2\pi i}\left[\frac{2}{1-\zeta^2}\int_L \frac{(1-\tau^2)f(\tau)d\tau}{\tau-\zeta} + 2\pi i\, f(\zeta) + \lambda^2 + \frac{2\lambda(k_1+\zeta k_o) - k_o^2}{1-\zeta^2}\right]^{\frac{1}{2}}$$

$$- \frac{1}{2\pi i}\left[\frac{2}{1-\zeta^2}\int_L \frac{(1-\tau^2)f(\tau)d\tau}{\tau-\zeta} - 2\pi i\, f(\zeta) + \lambda^2 + \frac{2\lambda(k_1+\zeta k_o) - k_o^2}{1-\zeta^2}\right]^{\frac{1}{2}} .$$

For the case in which $L[-1,1]$ coincides with the real axis we find that if $f(\xi)$ is real then the solution of

$$(2.13) \qquad \lambda\phi(\xi) + \phi(\xi)\int_{-1}^{1} \frac{\phi(t)dt}{t-\xi} = f(\xi)\,, \qquad -1 < \xi < 1$$

is

$$(2.14) \qquad \phi(\xi) = \frac{1}{\pi}\mathcal{I}\left[\frac{2}{1-\xi^2}\int_{-1}^{1}\frac{(1-t^2)f(t)dt}{t-\xi} + 2\pi i\ f(\xi) + \lambda^2 + \frac{1}{1-\xi^2}\left\{2\lambda(k_1+\xi k_o) - k_o^2\right\}\right]^{\frac{1}{2}} .$$

It is easy to verify that (2.12) satisfies (2.1) when S(z) satisfies condition (A). If we write

$$\phi(\zeta) = \frac{1}{2\pi i}\left[S_o(\zeta) + 2\pi i\ f(\zeta)\right]^{\frac{1}{2}} - \frac{1}{2\pi i}\left[S_o(\zeta) - 2\pi i\ f(\zeta)\right]^{\frac{1}{2}}$$

we have

$$\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} = \frac{1}{2\pi i}\int_L \frac{\left\{\left[S_o(\tau)+2\pi i\ f(\tau)\right]^{\frac{1}{2}} - \left[S_o(\tau)-2\pi i\ f(\tau)\right]^{\frac{1}{2}}\right\}d\tau}{\tau-\zeta} ,$$

$$\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} = \frac{1}{2\pi i}\oint_C \frac{S(\tau)d\tau}{\tau-\zeta} .$$

Now if we detach C from L and expand C into Γ we have

$$\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} = \frac{1}{2}\left\{\left[S_o(\zeta)+2\pi i\ f(\zeta)\right]^{\frac{1}{2}} + \left[S_o(\zeta)-2\pi i\ f(\zeta)\right]^{\frac{1}{2}}\right\} + \frac{1}{2\pi i}\oint_\Gamma \frac{S(\tau)d\tau}{\tau-\zeta}$$

and, since $S(\tau) \longrightarrow \lambda$ as $\tau \longrightarrow \infty$,

$$\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} = \frac{1}{2}\left\{\left[S_o(\zeta)+\ 2\pi i\ f(\zeta)\right]^{\frac{1}{2}} + \left[S_o(\zeta)-2\pi i\ f(\zeta)\right]^{\frac{1}{2}}\right\} - \lambda .$$

Therefore

$$\lambda\phi(\zeta)+\phi(\zeta)\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} = \frac{\lambda}{2\pi i}\left[S^+(\zeta)-S^-(\zeta)\right]$$

$$+\frac{1}{4\pi i}\left[S_o(\zeta)+2\pi i\ f(\zeta)-S_o(\zeta)+2\pi i\ f(\zeta)\right]$$

$$-\frac{\lambda}{2\pi i}\left[S^+(\zeta)-S^-(\zeta)\right]$$

$$= f(\zeta)\ .$$

The condition that S(z) must be analytic in the complex plane slit along L[-1,1], [the condition (A)], does not in general determine the constants k_o and k_1 numerically. This can be seen by considering the equation

$$\lambda\phi(\xi)+\phi(\xi)\int_{-1}^{1}\frac{\phi(t)dt}{t-\xi} = 0\ . \tag{2.15}$$

The function S(z) for this equation is

$$S(z) = \left[\lambda^2 + \frac{k_o^2 - 2\lambda k_1 - 2\lambda k_o z}{z^2-1}\right]^{\frac{1}{2}}$$

$$= \left[\frac{\lambda^2 z^2 - 2\lambda k_o z + k_o^2 - 2\lambda k_1 - \lambda^2}{z^2-1}\right]^{\frac{1}{2}}$$

and this certainly satisfies condition (A) if we take $k_o = k_1 = 0$. For these values we have

$$\phi(\xi) = \frac{1}{\pi}\mathcal{I}(\lambda^2)^{\frac{1}{2}} = 0\ ,$$

namely the trivial solution of (2.15). On the other hand, S(z) will satisfy (A) if

$$\lambda^2 z^2 - 2\lambda k_o z + k_o^2 - 2\lambda k_1 - \lambda^2$$

possesses a double zero, that is, if we take $2k_1 + \lambda = 0$. With this choice, k_o is arbitrary, and

$$S(z) = \frac{\lambda z - k_o}{\sqrt{z^2 - 1}}$$

from which

$$\phi(\xi) = \frac{1}{\pi} \mathscr{J} \frac{\lambda\xi - k_o}{\sqrt{\xi^2 - 1}} = \frac{1}{\pi} \cdot \frac{\lambda\xi - k_o}{\sqrt{1 - \xi^2}}$$

and this is the general solution of

$$\lambda + \int_{-1}^{1} \frac{\phi(t)dt}{t - \xi} = 0 .$$

Thus we can see that in order to fix k_o and k_1 uniquely we may have to impose side conditions in addition to condition (A).

Let us examine the possibility of having a solution ϕ whose endpoint behavior is such that

$$\text{(B)} \qquad \mathscr{L}_{\tau \to \alpha} (\alpha - \tau)^{\frac{1}{2}} \phi(\tau) = 0 .$$

With this behavior the endpoint behavior of $f(\zeta)$ must be such that

$$(\text{B}_1) \qquad \mathscr{L}_{\zeta \to \alpha} (\alpha - \zeta) f(\zeta) = 0 .$$

Under condition (B) the order of integration can be changed in

$$\int_L \phi(\zeta) \int_L \frac{\phi(\tau)d\tau d\zeta}{\tau - \zeta}$$

and hence by integrating (2.1) we find

$$\lambda\int_L \phi(\zeta)d\zeta - \int_L \phi(\tau)\int_L \frac{\phi(\zeta)d\zeta d\tau}{\zeta-\tau} = \int_L f(\zeta)d\zeta$$

$$\lambda\int_L \phi(\zeta)d\zeta + \int_L [\lambda\phi(\tau) - f(\tau)]d\tau = \int_L f(\zeta)d\zeta$$

(2.16) $$\lambda k_o = \int_L f(\tau)d\tau \ .$$

Also, after multiplying (2.1) by ζ and integrating:

$$\lambda\int_L \zeta\phi(\zeta)d\zeta - \int_L \phi(\tau)\int_L \frac{(\zeta-\tau+\tau)\phi(\zeta)d\zeta d\tau}{\zeta-\tau} = \int_L \zeta f(\zeta)d\zeta$$

$$\lambda\int_L \zeta\phi(\zeta)d\zeta - \left[\int_L \phi(\tau)d\tau\right]^2 + \int_L \tau[\lambda\phi(\tau) - f(\tau)]d\tau = \int_L \zeta f(\zeta)d\zeta$$

(2.17) $$2\lambda k_1 - 2\int_L \tau f(\tau)d\tau = k_o^2 \ .$$

The substitution of these values for k_o and k_1 in (2.9) gives

(2.18) $$S(z) = \left[2\int_L \frac{f(\tau)d\tau}{\tau - z} + \lambda^2\right]^{\frac{1}{2}}$$

and a solution satisfying (B) will exist if $f(\zeta)$ satisfies (B_1) and (2.18) satisfies (A). When this is the case the solution is

$$\phi(\zeta) = \frac{1}{2\pi i}\left[2\int_L \frac{f(\tau)d\tau}{\tau-\zeta} + 2\pi i\ f(\zeta) + \lambda^2\right]^{\frac{1}{2}}$$

$$- \frac{1}{2\pi i}\left[2\int_L \frac{f(\tau)d\tau}{\tau-\zeta} - 2\pi i\ f(\zeta) + \lambda^2\right]^{\frac{1}{2}} \ .$$

Instead of imposing the condition (B) at each endpoint of L we may wish to impose it at only one endpoint, say the condition

$$\text{(C)} \qquad \mathcal{L}_{\tau\to 1}(1-\tau)^{\frac{1}{2}}\phi(\tau) = 0 .$$

The function $f(\zeta)$ would then have to be such that

$$(C_1) \qquad \mathcal{L}_{\zeta\to 1}(1-\zeta)f(\zeta) = 0 .$$

Under condition (C) we find from (2.1) that

$$\lambda\int_L (1+\zeta)\phi(\zeta)d\zeta - \int_L \phi(\tau)\int_L \frac{(\zeta-\tau+\tau+1)\phi(\zeta)d\zeta d\tau}{\zeta-\tau} = \int_L (1+\zeta)f(\zeta)d\zeta$$

$$\lambda\int_L (1+\zeta)\phi(\zeta)d\zeta - \left[\int_L \phi(\tau)d\tau\right]^{\frac{1}{2}} + \int_L (1+\tau)[\lambda\phi(\tau)-f(\tau)]d\tau = \int_L (1+\zeta)f(\zeta)d\zeta$$

$$\text{(2.19)} \qquad 2\lambda(k_o+k_1) - k_o^2 = 2\int_L (1+\tau)f(\tau)d\tau .$$

The substitution of this value of k_1 in (2.9) gives

$$\text{(2.20)} \qquad S'(z) = \left[\frac{2}{1+z}\int_L \frac{(1+\tau)f(\tau)d\tau}{\tau-z} - \frac{2\lambda k_o}{1+z} + \lambda^2\right]^{\frac{1}{2}}$$

and a solution satisfying (C) will exist if $f(\zeta)$ satisfies (C_1) and (2.20) satisfies (A). When this is the case the solution is

$$\phi(\zeta) = \frac{1}{2\pi i}\left[\frac{2}{1+\zeta}\int_L \frac{(1+\tau)f(\tau)d\tau}{\tau-\zeta} + 2\pi i\, f(\zeta) - \frac{2\lambda k_o}{1+\zeta} + \lambda^2\right]^{\frac{1}{2}}$$

$$- \frac{1}{2\pi i}\left[\frac{2}{1+\zeta}\int_L \frac{(1+\tau)f(\tau)d\tau}{\tau-\zeta} - 2\pi i\, f(\zeta) - \frac{2\lambda k_o}{1+\zeta} + \lambda^2\right]^{\frac{1}{2}} .$$

3. Equation II

In this section, we show that

$$(3.1) \qquad \pi^2\phi^2(\zeta) - \left[\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta}\right]^2 = f(\zeta)$$

can be converted into a particular case of (2.1). We saw in Section 2 that if $\phi(\tau)$ belongs to the class $\mathcal{U}_1$ then we have the identity

$$(3.2) \qquad \frac{2}{1-\omega^2}\int_L \frac{(1-\zeta^2)\phi(\zeta)}{\zeta-\omega}\int_L \frac{\phi(\tau)d\tau d\zeta}{\tau-\zeta}$$

$$= -\pi^2\phi^2(\omega) + \left[\int_L \frac{\phi(\tau)d\tau}{\tau-\omega}\right]^2 + \frac{1}{1-\omega^2}\left[\int_L \phi(\tau)d\tau\right]^2 .$$

Hence if $\phi(\tau)$ satisfies (3.1) then it must satisfy

$$(3.3) \qquad \frac{2}{1-\omega^2}\int_L \frac{(1-\zeta^2)\phi(\zeta)}{\zeta-\omega}\int_L \frac{\phi(\tau)d\tau d\zeta}{\tau-\zeta} = -f(\omega) + \frac{k_o^2}{1-\omega^2}$$

or

$$(3.4) \qquad 2\int_L \frac{(1-\zeta^2)\phi(\zeta)}{\zeta-\omega}\int_L \frac{\phi(\tau)d\tau d\zeta}{\tau-\zeta} = -(1-\omega^2)f(\omega) + k_o^2$$

where

$$k_o = \int_L \phi(\tau)d\tau .$$

Equation (3.4) is a singular integral equation of the first kind which can be solved for

$$2(1-\zeta^2)\phi(\zeta)\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} .$$

The solution is

$$(3.5)\qquad 2(1-\zeta^2)\phi(\zeta)\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta}$$

$$= -\frac{1}{\pi^2\sqrt{1-\zeta^2}}\int_L \frac{\sqrt{1-\tau^2}\,[k_o^2-(1-\tau^2)f(\tau)]d\tau}{\tau-\zeta} + \frac{k}{\sqrt{1-\zeta^2}}$$

where

$$\pi k = 2\int_L (1-\zeta^2)\phi(\zeta)\int_L \frac{\phi(\tau)d\tau d\zeta}{\tau-\zeta}$$

$$= -2\int_L (1-\tau^2)\phi(\tau)\int_L \frac{\phi(\zeta)d\zeta d\tau}{\zeta-\tau} + 2\int_L \phi(\tau)\int_L (\zeta+\tau)\phi(\zeta)d\zeta d\tau$$

and therefore

$$\pi k = \int_L \phi(\tau)\int_L (\zeta+\tau)\phi(\zeta)d\zeta d\tau$$

$$(3.6)\qquad \pi k = 2k_o k_1 \quad .$$

From (3.5) and (3.6) we find that if $\phi(\tau)$ satisfies (3.1) it must satisfy

$$(3.7)\qquad \phi(\zeta)\int_L \frac{\phi(\tau)d\tau}{\tau-\zeta} = F(\zeta)$$

where

$$(3.8)\qquad F(\zeta) = \frac{1}{2\pi^2(1-\zeta^2)^{3/2}}\int_L \frac{(1-\tau^2)^{3/2}f(\tau)d\tau}{\tau-\zeta}$$

$$+ \frac{\zeta k_o^2}{2\pi(1-\zeta^2)^{3/2}} + \frac{k_o k_1}{\pi(1-\zeta^2)^{3/2}} \quad .$$

Conversely, if $\phi(\tau)$ satisfies (3.7) it must satisfy (3.1).

Equation (3.7) is equation (2.1) with $\lambda = 0$. We can apply the theory of Section 2. It follows from the results of Section 2 that (3.1) possesses a solution provided

$$S_1(z) = \left[\frac{1}{(1-z^2)} \int_L \frac{(1-\tau^2)F(\tau)d\tau}{\tau - z} - \frac{k_o^2}{1-z^2}\right]^{\frac{1}{2}}$$

satisfies condition (A), i.e., $S_1(z)$ is analytic for z not on L. Since

$$\frac{2}{1-z^2} \int_L \frac{(1-\tau^2)F(\tau)d\tau}{\tau - z} = \frac{1}{\pi^2(1-z^2)} \int_L \frac{1}{(\tau-z)\sqrt{1-\tau^2}} \int_L \frac{(1-\zeta^2)^{3/2}f(\zeta)d\zeta d\tau}{\zeta - \tau}$$

$$+ \frac{k_o^2}{\pi(1-z^2)} \int_L \frac{\tau d\tau}{(\tau-z)\sqrt{1-\tau^2}} + \frac{2k_o k_1}{\pi(1-z^2)} \int_L \frac{d\tau}{(\tau-z)\sqrt{1-\tau^2}}$$

$$= \frac{1}{\pi(1-z^2)^{3/2}} \int_L \frac{(1-\zeta^2)^{3/2}f(\zeta)d\zeta}{\zeta - z}$$

$$+ \frac{k_o^2}{(1-z^2)} + \frac{ik_o(2k_1 + k_o z)}{(1-z^2)^{3/2}}$$

we see that if (3.1) possesses a solution then the constants k_o and k_1 must be such that

$$S_1(z) = \left[\frac{1}{\pi(1-z^2)^{3/2}} \int_L \frac{(1-\zeta^2)^{3/2}f(\zeta)d\zeta}{\zeta - z} + \frac{ik_o(2k_1 + k_o z)}{(1-z^2)^{3/2}}\right]^{\frac{1}{2}}$$

is analytic for z in the exterior of L. If this condition can be satisfied by a proper choice of k_o and k_1, the solution of (3.1) is

$$(3.9) \qquad \phi(\tau) = \frac{1}{2\pi i}\left[- f(\tau) + \frac{i}{\pi^2(1-\tau^2)^{3/2}} \int_L \frac{(1-\zeta^2)^{3/2} f(\zeta)\,d\zeta}{(\zeta-\tau)} + \frac{ik_o(2k_1+k_o\tau)}{(1-\tau^2)^{3/2}}\right]^{\frac{1}{2}}$$

$$- \frac{1}{2\pi i}\left[- f(\tau) - \frac{i}{\pi^2(1-\tau^2)^{3/2}} \int_L \frac{(1-\zeta^2)^{3/2} f(\zeta)\,d\zeta}{\zeta-\tau} - \frac{ik_o(2k_1+k_o\tau)}{(1-\tau^2)^{3/2}}\right]^{\frac{1}{2}}$$

For the case in which $L[-1,1]$ coincides with the real axis we find that if $f(\zeta)$ is real then the real solution of

$$(3.10) \qquad \pi^2\phi^2(\xi) - \left[\int_{-1}^{1} \frac{\phi(t)\,dt}{t-\xi}\right]^2 = f(\xi)\,, \qquad -1 < \xi < 1$$

is

$$(3.11) \qquad \phi(\xi) = \frac{1}{\pi}\mathcal{J}\left[- f(\xi) + \frac{i}{\pi(1-\xi^2)^{3/2}} \int_{-1}^{1} \frac{(1-t^2)^{3/2} f(t)\,dt}{t-\xi} + \frac{ik_o(2k_1+k_o\xi)}{(1-\xi^2)^{3/2}}\right]^{\frac{1}{2}}$$

provided condition (A) is satisfied.

4. Equation III

The equation

$$(4.1) \qquad \pi^2\phi^2(\zeta) + \left[\int_L \frac{\phi(\tau)\,d\tau}{\tau-\zeta}\right]^2 = f(\zeta)$$

can be written

$$\left[\pi i\ \phi(\zeta) + \int_L \frac{\phi(\tau)d\tau}{\tau - \zeta}\right] \cdot \left[-\pi i\ \phi(\zeta) + \int_L \frac{\phi(\tau)d\tau}{\tau - \zeta}\right] = f(\zeta) \ . \tag{4.2}$$

We will solve this equation by using Carleman's method. Let us introduce the function

$$F(z) = \int_L \frac{\phi(\tau)d\tau}{\tau - z} \ , \tag{4.3}$$

a function of the complex variable $z = x + iy$, which is analytic for z not on L, a simple smooth arc directed from $z = -1$ to $z = 1$. If we let z approach a point ζ on L from the positive side of L, and then the negative side of L; the limit values of $F(z)$ are

$$F^+(\zeta) = \pi i\ \phi(\zeta) + \int_L \frac{\phi(\tau)d\tau}{\tau - \zeta} \tag{4.4}$$

$$F^-(\zeta) = -\pi i\ \phi(\zeta) + \int_L \frac{\phi(\tau)d\tau}{\tau - \zeta} \ . \tag{4.5}$$

Equation (4.2) says that these limit values must satisfy the barrier equation

$$F^+(\zeta) \cdot F^-(\zeta) = f(\zeta) \tag{4.6}$$

where ζ is not an endpoint of L. The function $F(z)$ which satisfies this equation and has the properties implied by the representation (4.3) provides the solution of the integral equation because the subtraction of the Plemelj formulas (4.4) and (4.5) gives

$$\phi(\zeta) = \frac{1}{2\pi i}\left[F^+(\zeta) - F^-(\zeta)\right] \ . \tag{4.7}$$

If we write

$$(4.8) \qquad F(z) = \frac{e^{G(z)}}{z + i\sqrt{1-z^2}}$$

then (4.6) implies that $G(z)$ must satisfy

$$(4.9) \qquad G^+(\zeta) + G^-(\zeta) = \ln f(\zeta) + 2\pi i\, n$$

where n is an integer. As can be seen by using the Plemelj formulas, the function

$$(4.10) \qquad G_1(z) = \frac{\sqrt{1-z^2}}{2\pi i} \int_L \frac{\ln f(\tau)\,d\tau}{\left(\sqrt{1-\tau^2}\right)^+(\tau - z)} + \pi i\, n$$

is a particular solution of (4.9). By $\sqrt{1-z^2}$ we mean here, and in what follows, the branch of the two valued function $(1-z^2)^{1/2}$ which is analytic for any finite z not on L, and such that

$$\mathcal{L}_{z\to\infty} \frac{\sqrt{1-z^2}}{z} = -i \ .$$

The function $G_1(z)$ is analytic for z not on L and since

$$\mathcal{L}_{z\to\infty} G_1(z) = \text{const.} \ ,$$

we have

$$\mathcal{L}_{z\to\infty} \frac{e^{G_1(z)}}{z + i\sqrt{1-z^2}} = 0 \ .$$

The general solution of (4.9) is

$$G(z) = G_1(z) + \sqrt{1-z^2}\, p(z)$$

where $p(z)$ must satisfy

$$p^+(\zeta) - p^-(\zeta) = 0 \;. \tag{4.11}$$

The function p(z) must be taken so that the properties of $e^{G(z)}/ z + i\sqrt{1-z^2}$ match those of

$$F(z) = \int_L \frac{\phi(\tau)d\tau}{\tau - z} \;.$$

This function is analytic for z not on L and it vanishes like c_o/z as $z \to \infty$, provided $c_o = \int_L \phi(\tau)d\tau \neq 0$. Furthermore, in accordance with the assumptions about $\phi(\tau)$ admitted in the introduction, the behavior of F(z) in the neighborhood of an endpoint α of L is such that $\mathcal{L}_{z\to\alpha}(\alpha-z)F(z) = 0$; and the limit values $F^+(\zeta)$, $F^-(\zeta)$ must satisfy a uniform Hölder condition. These properties and the condition (4.11) imply that p(z) must be analytic everywhere and it must vanish at infinity, i.e., p(z) = 0.

We have now found that

$$F(z) = \pm\left(z - i\sqrt{1-z^2}\right)\cdot \exp\left\{\frac{\sqrt{1-z^2}}{2\pi i}\int_L \frac{\ln f(\tau)d\tau}{\left(\sqrt{1-\tau^2}\right)^+(\tau-z)}\right\}. \tag{4.12}$$

This gives

$$F^+(\zeta) = \pm\left(\zeta - i\sqrt{1-\zeta^2}\right)\sqrt{f(\zeta)}\;.\exp\left\{\frac{\sqrt{1-\zeta^2}}{2\pi i}\int_L \frac{\ln f(\tau)d\tau}{\sqrt{1-\tau^2}(\tau-\zeta)}\right\} \tag{4.13}$$

$$F^-(\zeta) = \pm\left(\zeta + i\sqrt{1-\zeta^2}\right)\sqrt{f(\zeta)}\;.\exp\left\{-\frac{\sqrt{1-\zeta^2}}{2\pi i}\int_L \frac{\ln f(\tau)d\tau}{\sqrt{1-\tau^2}(\tau-\zeta)}\right\}. \tag{4.14}$$

The solution of (4.1) is obtained by subtracting (4.14) from (4.13). It is

$$(4.15)\quad \phi(\zeta) = \frac{1}{2\pi i}\left[F^+(\zeta) - F^-(\zeta)\right]$$

$$= \pm\frac{\sqrt{f(\zeta)}}{\pi}\left\{\begin{array}{l}\zeta\,\sin\left[\frac{\sqrt{1-\zeta^2}}{2\pi}\int\limits_L \frac{\ln f(\tau)d\tau}{\sqrt{1-\tau^2}(\tau-\zeta)}\right]\\ \\ +\sqrt{1-\zeta^2}\,\cos\left[\frac{\sqrt{1-\zeta^2}}{2\pi}\int\limits_L \frac{\ln f(\tau)d\tau}{\sqrt{1-\tau^2}(\tau-\zeta)}\right]\end{array}\right\}.$$

It will be noticed that in the above analysis we have assumed that $F(z) = \int\limits_L \frac{\phi(\tau)d\tau}{\tau - z}$ vanishes like c_o/z, $c_o \neq 0$, as $z \to \infty$. If this is not the case, that is, if for example

$$(4.16)\qquad c_o = -\int\limits_L \phi(\tau)d\tau = 0$$

while

$$(4.17)\qquad c_1 = -\int\limits_L \tau\phi(\tau)d\tau = 0$$

then $F(z)$ vanishes like c_1/z^2, $c_1 \neq 0$, as $z \to \infty$ and the solution of (4.6) as we have given it has to be adjusted. However, it is clear from the above analysis that if (4.16) and (4.17) prevail then the adjustment is easily made by taking $F(z) = e^{G(z)}/\left(z + i\sqrt{1-z^2}\right)^2$ instead of (4.8), which leads to

$$(4.18)\quad F(z) = \pm\left[z - i\sqrt{1-z^2}\right]^2 \exp\left\{\frac{\sqrt{1-z^2}}{2\pi i}\int\limits_L \frac{\ln f(\tau)d\tau}{\left(\sqrt{1-\tau^2}\right)^+(\tau-z)}\right\}.$$

5. Some Non-linear Problems in Potential Theory

The equations we have analyzed can be identified with certain problems in potential theory. For example, consider the problem of finding $\psi(x,y)$ such that

$$\psi_{xx}(x,y) + \psi_{yy}(x,y) = 0 , \qquad y < 0 ,$$

$$\psi_y(x,0) = 0 , \qquad |x| > 1$$

and subject to an additional condition on $y = 0$, $|x| < 1$ which is specified below. Let us suppose that $\psi_y(x,0)$, $|x| < 1$, satisfies the conditions imposed on $\phi(t)$ in Section 1, and that each of $\psi_x(x,y)$ and $\psi_y(x,y)$ vanishes as $z = x + iy \rightarrow \infty$. The harmonic function $\psi_y(x,y)$ can then be written

$$\psi_y(x,y) = -\frac{1}{\pi}\int_{-1}^{1} \frac{y\psi_y(t,0)dt}{(t-x)^2+y^2}$$

$$= -\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{-1}^{1} \ln[(t-x)^2+y^2]\psi_y(t,0)dt$$

from which

$$\psi_{xy}(x,y) = \frac{1}{\pi}\frac{\partial}{\partial y}\int_{-1}^{1} \frac{(t-x)\psi_y(t,0)dt}{(t-x)^2+y^2}$$

and by integration

$$\psi_x(x,y) = \frac{1}{\pi}\int_{-1}^{1} \frac{(t-x)\psi_y(t,0)dt}{(t-x)^2+y^2}$$

$$\psi_x(x,0) = \frac{1}{\pi}\int_{-1}^{1} \frac{\psi_y(t,0)dt}{t-x} .$$

Now if we impose the additional condition

$$\text{I} \qquad \frac{\lambda}{\pi}\,\psi_y(x,0) + \psi_x(x,0)\psi_y(x,0) = \frac{f(x)}{\pi}\,, \qquad |x| < 1$$

then $\phi(t) = \psi_y(t,0)$, $|t| < 1$, must satisfy

$$\lambda\phi(x) + \phi(x)\int_{-1}^{1}\frac{\phi(t)dt}{t-x} = f(x)\,, \qquad |x| < 1\,.$$

If instead of I we impose

$$\text{II} \qquad \psi_y^2(x,0) - \psi_x^2(x,0) = \frac{f(x)}{\pi^2}\,, \qquad |x| < 1$$

then $f(t) = \psi_y(t,0)$, must satisfy

$$\pi^2\phi^2(x) - \left[\int_{-1}^{1}\frac{\phi(t)dt}{t-x}\right]^2 = f(x)\,, \qquad |x| < 1\,.$$

Finally, if the additional condition is

$$\text{III} \qquad \psi_y^2(x,0) + \psi_x^2(x,0) = \frac{f(x)}{\pi^2}\,, \qquad |x| < 1$$

then $\phi(t) = \psi_y(t,0)$, must satisfy

$$\pi^2\phi^2(x) + \left[\int_{-1}^{1}\frac{\phi(t)dt}{t-x}\right]^2 = f(x)\,, \qquad |x| < 1\,.$$

We proceed to show that problem III can be solved for domains more general than the half plane. For the half plane problem we can write

$$\psi(x,y) = \mathcal{R}\,\mathcal{F}(z)$$

and then

$$\mathfrak{F}'(z) = \mathcal{R}\mathfrak{F}'(z) + i\,\mathcal{I}\mathfrak{F}'(z) = \psi_x(x,y) - i\psi_y(x,y)$$

$$= \frac{1}{\pi}\,\mathcal{R}\int_{-1}^{1} \frac{\phi(t)dt}{t-z} + \frac{i}{\pi}\,\mathcal{I}\int_{-1}^{1} \frac{\phi(t)dt}{t-z}$$

which is the same as

$$\mathfrak{F}'(z) = \frac{1}{\pi}\int_{-1}^{1} \frac{\phi(t)dt}{t-z} \; . \tag{5.1}$$

The integral on the right hand side of (5.1) has been determined in connection with the problem of solving the integral equation III, in Section 4, and we have

$$\mathfrak{F}'(z) = \pm\frac{(z - i\sqrt{1-z^2})^{\gamma}}{\pi} \exp\left\{\frac{\sqrt{1-z^2}}{2\pi i} \int_{-1}^{1} \frac{\ln f(t)dt}{\left(\sqrt{1-t^2}\right)^{+}(t-z)}\right\} \tag{5.2}$$

where γ is 1, or 2. The function $\psi(x,y)$ can be obtained from (5.2) by an integration.

Let $z = m(\zeta) = m(\xi + i\eta)$ be a function which maps the domain D, with boundary C in the ζ-plane, conformally into the lower half of the z-plane. Let the image of C_1 a part of C, be the segment $y = 0$, $|x| < 1$; and let the image of C_2, the remaining part of C, be $y = 0$, $|x| > 1$. Under this mapping, $\mu(x,y)$ is transformed into

$$\psi(x,y) = \underline{\Psi}(\xi,\eta) = \mathcal{R}\mathfrak{F}[m(\zeta)] \; ,$$

a function harmonic in D.

Let s be the arc length measured along C from say the initial point of C_1. If $\zeta = \xi(s) + i\eta(s)$ is a point on C, then the normal derivative of $\underline{\Psi}(\xi,\eta)$ at the boundary is

$$\text{(5.3)} \qquad \frac{\partial \Psi(\xi,\eta)}{\partial n} = \frac{\partial}{\partial n} \mathcal{R}\,\mathfrak{F}[m(\sigma)]$$

$$= \frac{\partial}{\partial s} \mathcal{I}\,\mathfrak{F}[m(\sigma)]$$

$$= \mathcal{I}\,\frac{\partial m(\sigma)}{\partial s}\,\mathfrak{F}'[m(\sigma)] \;.$$

If σ is on C_2, then $\frac{dm(\sigma)}{ds} = \frac{dx}{ds}$ is real and $\mathfrak{F}'[m(\sigma)]$ is real. Hence for σ on C_2

$$\Psi_n(\xi,\eta) = 0 \;.$$

The tangential derivative of $\Psi(\xi,\eta)$ at the boundary is

$$\text{(5.4)} \qquad \frac{\partial \Psi(\xi,\eta)}{\partial s} = \mathcal{R}\,\frac{\partial m(\sigma)}{\partial s}\,\mathfrak{F}'[m(\sigma)] \;.$$

From (5.3) and (5.4) we see that for σ on C_1

$$\Psi_n^2(\xi,\eta) + \Psi_s^2(\xi,\eta) = \left|\frac{dm(\sigma)}{ds}\right|^2 \left|\mathfrak{F}'[m(\sigma)]\right|^2$$

$$= \left|\frac{dm(\sigma)}{ds}\right|^2 \left[\{\mathcal{R}\,\mathfrak{F}'[m(\sigma)]\}^2 + \{\mathcal{I}\,\mathfrak{F}'[m(\sigma)]\}^2\right]$$

$$= \left|\frac{dm(\sigma)}{ds}\right|^2 \cdot \frac{f[m(\sigma)]}{\pi^2} \;.$$

Therefore if we take

$$h(s) = \left|\frac{dm(\sigma)}{ds}\right|^2 \frac{f[m(\sigma)]}{\pi^2}$$

as prescribed, then

$$\Psi(\xi,\eta) = \mathcal{R}\,\mathfrak{F}[m(\zeta)]$$

is harmonic in D, with normal derivative equal to zero along C_2, and with

$$\Psi_n^2(\xi,\eta) + \Psi_s^2(\xi,\eta) = h(s)$$

prescribed along C_1. Thus the solution of problem III for the domain D can be found by substituting $z = m(\zeta)$, $t = m(\sigma)$ and

$$f(t) = f[m(\sigma)] = \frac{\pi^2 h(s)}{\left|\frac{dm(\sigma)}{ds}\right|^2}$$

in (5.2). The substitution gives

$$(5.5)\quad \frac{d\mathfrak{F}[m(\zeta)]}{d\zeta} = \pm \frac{m'(\zeta)}{\pi}\left[m(\zeta) - i\sqrt{1-m^2(\zeta)}\right]^{\gamma}$$

$$\cdot \exp\left\{-\frac{\sqrt{1-m^2(\zeta)}}{2\pi i}\int_{C_1} \frac{\ln\left\{\left|\frac{\pi}{\frac{dm(\sigma)}{ds}}\right|^2 \cdot h(s)\right\}\cdot \frac{dm(\sigma)}{ds}\,ds}{\sqrt{1-m^2(\sigma)}[m(\sigma) - m(\zeta)]}\right\}$$

and then $\Psi(\xi,\eta)$ can be obtained by integration.

The solution of problem III for particular domains and special boundary values of the gradient is not new. For example, it is well known that the method of conformal mapping can be used to find a potential function $\Psi(\xi,\eta)$ such that

$$\Psi_{\xi\xi}(\xi,\eta) + \Psi_{\eta\eta}(\xi,\eta) = 0\,, \qquad -\infty < \xi < \infty\;; \quad -1 < \eta < 1\;;$$

$$\psi_n(\xi,\eta) = 0 \qquad \eta = -1,1\;; \quad -\infty < \xi < 0\;;$$

$$\psi_\xi^2(\xi,\eta) + \psi_\eta^2(\xi,\eta) = \text{const.} \qquad \eta = -1,1\;; \quad 0 < \xi < \infty\,.$$

This problem, or its equivalent, occurs in the hydrodynamical theory of jets. However, the author has not seen the formula (5.5) in the literature and it is probable that there are several problems in field or flow theory for which it would be advantageous to use (5.5) directly.

In order to see a simple application of (5.5) consider the problem of finding $\Psi(\xi,\eta)$ such that

$$\Psi_{\xi\xi}(\xi,\eta) + \Psi_{\eta\eta}(\xi,\eta) = 0 \,, \qquad -\infty < \xi < \infty \;; \quad 0 < \eta < 1$$

$$(5.6) \qquad \Psi_{\eta}(\xi,0) = 0 \,, \qquad -\infty < \xi < \infty$$

$$\Psi_{\xi}^{2}(\xi,1) + \Psi_{\eta}^{2}(\xi,1) = h(\xi) \,, \qquad -\infty < \xi < \infty$$

The function

$$(5.7) \qquad z = x + iy = \coth \frac{\pi}{2}(\xi + i\eta) = \coth \frac{\pi}{2}\zeta = m(\zeta)$$

maps the infinite strip into the lower half of the z-plane and the segment $\mathcal{I} z = 0$; $|\mathcal{R}(z)| < 1$ is the image of C_1: $\zeta = \xi + i$; $-\infty < \xi < \infty$. From (5.7) we find

$$m'(\zeta) = -\frac{\pi}{2} \operatorname{csch}^2 \frac{\pi}{2}\zeta$$

$$\sqrt{1 - m^2(\zeta)} = \sqrt{1 - \coth^2 \frac{\pi\zeta}{2}} = -i \operatorname{csch} \frac{\pi\zeta}{2}$$

$$m(\sigma) = m(\lambda + i) = \tanh \frac{\pi}{2}\lambda$$

$$\sqrt{1 - m^2(\sigma)} = \sqrt{1 - \tanh^2 \frac{\pi\lambda}{2}} = \operatorname{sech} \frac{\pi}{2}\lambda$$

$$\frac{dm(\sigma)}{ds} = \frac{d}{d\lambda}\left(\tanh \frac{\pi}{2}\lambda\right)\frac{d\lambda}{ds} \,, \qquad \frac{d\lambda}{ds} = -1$$

$$= -\frac{\pi}{2} \operatorname{sech}^2 \frac{\pi}{2}\lambda \,,$$

$$\frac{1}{m(\sigma) - m(\zeta)} = -\frac{\cosh \frac{\pi}{2}\lambda \cdot \sinh \frac{\pi}{2}\zeta}{\cosh \frac{\pi}{2}(\lambda - \zeta)} \,.$$

The substitution of these quantities in (5.5) gives

$$(5.8)\quad \frac{d}{d\zeta}\mathfrak{F}[m(s)] = \pm\frac{1}{2\sinh^2\frac{\pi}{2}\zeta}\left[\frac{\cosh\frac{\pi}{2}\zeta - 1}{\sinh\frac{\pi}{2}\zeta}\right]^{\gamma}$$

$$\cdot\ \exp\left\{\frac{1}{4}\int_{-\infty}^{\infty}\frac{\ln[4\cdot\cosh^4\frac{\pi}{2}\lambda\cdot h(\lambda)]d\lambda}{\cosh\frac{\pi}{2}(\lambda-\zeta)}\right\}$$

which if we use

$$(5.9)\quad \frac{1}{4}\int_{-\infty}^{\infty}\frac{\ln[4\cosh^4\frac{\pi}{2}\lambda]d\lambda}{\cosh\frac{\pi}{2}(\lambda-\zeta)}$$

$$= \ln 2 + 2\ln[1+\cosh\frac{\pi}{2}\zeta] + 2\pi i\eta$$

becomes

$$(5.10)\quad \frac{d}{d\zeta}\mathfrak{F}[m(\zeta)] = \pm\frac{2}{2\sinh^2\frac{\pi}{2}\zeta}\left[\frac{\cosh\frac{\pi}{2}\zeta-1}{\sinh\frac{\pi}{2}\zeta}\right]^{\gamma}\cdot\left[\cosh\frac{\pi}{2}\zeta+1\right]^2$$

$$\cdot\ \exp\ \frac{1}{4}\int_{-\infty}^{\infty}\frac{h(\lambda)d\lambda}{\cosh\frac{\pi}{2}(\lambda-\zeta)}\quad .$$

If we choose $\gamma = 2$ so as to avoid a singularity at $\zeta = 0$ we finally have

$$(5.11)\quad \frac{d}{d\zeta}\mathfrak{F}[m(\zeta)] = \pm\exp\left\{\frac{1}{4}\int_{-\infty}^{\infty}\frac{h(\lambda)d\lambda}{\cosh\frac{\pi}{2}(\lambda-\zeta)}\right\}$$

which determines the derivatives of $\Psi(\xi,\eta)$. That is, if

$$\Psi(\xi,\eta) = \mathcal{R}\, M(\zeta)$$

is required to satisfy (5.6) then

$$M'(\zeta) = \pm\exp\left\{\frac{1}{4}\int_{-\infty}^{\infty}\frac{h(\lambda)d\lambda}{\cosh\frac{\pi}{2}(\lambda-\zeta)}\right\}.$$

6. Some Non-linear Systems

In a recent paper, A. Beurling [5] proved that if D is a region in Euclidean n-space then the non-linear system

$$(6.1)\qquad \begin{aligned} \phi(x)\int_D k(t,x)\psi(t)dt &= f(x) \\ \psi(x)\int_D k(x,t)\phi(t)dt &= g(x) \end{aligned}$$

possesses a unique solution when the kernel k(t,x) and the prescribed functions f(x), g(x) are suitably restricted. Since Beurling's analysis does not cover the following singular one-dimensional case

$$(6.2)\qquad \begin{aligned} \phi(x)\int_{-1}^{1} \frac{\psi(t)dt}{t-x} &= f(x) \\ \psi(x)\int_{-1}^{1} \frac{\phi(t)dt}{t-x} &= g(x) \end{aligned} \qquad -1 < x < 1$$

and since this system is related to certain problems in potential theory; it is of some interest to see that (6.2) can be reduced to a linear system by using the method of Section 2.

We suppose that each of the unknown functions belongs to the class $\mathcal{H}_1$, defined in the introduction, and we suppose that each of the prescribed functions f(x), g(x) belongs to the class $\mathcal{H}_2$. Then, by using the Hardy-Poincaré-Bertrand formula as shown in Section 2 it is easy to verify that

$$(6.3)\quad \frac{1}{1-\xi^2}\int_{-1}^{1}\frac{(1-x^2)}{x-\xi}\left[\phi(x)\int_{-1}^{1}\frac{\psi(t)dt}{t-x}+\psi(x)\int_{-1}^{1}\frac{\phi(t)dt}{t-x}\right]dx-\frac{a_ob_o}{1-\xi^2}$$

$$= -\pi^2\phi(\xi)\psi(\xi)+\int_{-1}^{1}\frac{\psi(t)dt}{t-\xi}\cdot\int_{-1}^{1}\frac{\phi(t)dt}{t-\xi}$$

where

$$a_o = \int_{-1}^{1}\phi(t)dt$$

$$b_o = \int_{-1}^{1}\psi(t)dt\ .$$

If (6.2) possesses a solution, then (6.2) and (6.3) imply

$$(6.4)\quad \frac{1}{1-\xi^2}\int_{-1}^{1}\frac{(1-x^2)[f(x)+g(x)]dx}{x-\xi}-\frac{a_ob_o}{1-\xi^2}$$

$$= -\pi^2\phi(\xi)\psi(\xi)+\int_{-1}^{1}\frac{\psi(t)dt}{t-\xi}\cdot\int_{-1}^{1}\frac{\phi(t)dt}{t-\xi}$$

or after multiplying by $\phi(\xi)\psi(\xi)$

$$(6.5)\quad \pi^2\phi^2(\xi)\psi^2(\xi)+\frac{\phi(\xi)\psi(\xi)}{1-\xi^2}\left[\int_{-1}^{1}\frac{(1-x^2)[f(x)+g(x)]dx}{x-\xi}-a_ob_o\right]$$

$$-\phi(\xi)\int_{-1}^{1}\frac{\psi(t)dt}{t-\xi}\cdot\psi(\xi)\int_{-1}^{1}\frac{\phi(t)dt}{t-\xi}=0\ .$$

Using (6.2) again, equation (6.5) becomes

$$(6.6)\qquad \pi^2\phi^2(\xi)\psi^2(\xi)+\frac{\phi(\xi)\psi(\xi)}{1-\xi^2}F(\xi)-f(\xi)g(\xi)=0$$

where

$$F(\xi)=\int_{-1}^{1}\frac{(1-x^2)[f(x)+g(x)]dx}{x-\xi}-a_ob_o\ .$$

Equation (6.6) fixes the product $\phi(\xi)\psi(\xi)$. We find

$$\phi(\xi)\psi(\xi) = k(\xi) = -\frac{F(\xi) \pm \sqrt{F^2(\xi) + 4\pi^2 f(\xi)g(\xi)(1-\xi^2)^2}}{2\pi^2(1-\xi^2)} . \tag{6.7}$$

With this product known the system (6.2) can be written

$$\begin{aligned} \int_{-1}^{1} \frac{\psi(t)dt}{t-x} &= \frac{f(x)}{k(x)} \cdot \psi(x) , \\ \int_{-1}^{1} \frac{\phi(t)dt}{t-x} &= \frac{g(x)}{k(x)} \cdot \phi(x) . \end{aligned} \tag{6.8}$$

The last equations can now be solved by Carleman's method [4].

Notice that if each of f(x) and g(x) is integrable and we require each of the unknown functions to satisfy the condition (B) of Section 2 then

$$\begin{aligned} \int_{-1}^{1} f(x)dx &= \int_{-1}^{1} \phi(x) \int_{-1}^{1} \frac{\psi(t)dtdx}{t-x} = -\int_{-1}^{1} \psi(t) \int_{-1}^{1} \frac{\phi(x)dxdt}{x-t} \\ &= -\int_{-1}^{1} g(t)dt \end{aligned}$$

or

$$\int_{-1}^{1} [f(x) + g(x)]dx = 0 . \tag{6.9}$$

Also,

$$\begin{aligned} \int_{-1}^{1} xf(x)dx &= -\int_{-1}^{1} \psi(t) \int_{-1}^{1} \frac{(x-t+t)}{x-t} \phi(x)dxdt \\ &= -\int_{-1}^{1} \psi(t)dt \cdot \int_{-1}^{1} \phi(x)dx - \int_{-1}^{1} t\psi(t) \int_{-1}^{1} \frac{\phi(x)dxdt}{x-t} \end{aligned}$$

which determines $a_o b_o$, that is,

$$-a_o b_o = \int_{-1}^{1} x[f(x) + g(x)]dx .$$

Hence for solutions which satisfy (B), $F(\xi)/(1-\xi^2)$ becomes

$$\frac{F(\xi)}{1-\xi^2} = \int_{-1}^{1} \frac{[f(x) + g(x)]dx}{x - \xi} - \frac{\xi}{1-\xi^2} \int_{-1}^{1} [f(x) + g(x)]dx$$

and since the second term on the right is zero by (6.9) we have

$$\frac{F(\xi)}{1-\xi^2} = \int_{-1}^{1} \frac{[f(x) + g(x)]dx}{x - \xi} .$$

We remark in passing that the system

$$\phi_1(x) \int_{-1}^{1} \frac{\psi_1(t)dt}{t + x} = f_1(x)$$

$$-1 < x < 1$$

$$\psi_1(x) \int_{-1}^{1} \frac{\phi_1(t)dt}{t + x} = g_1(x)$$

can be reduced to (6.2) by using the substitutions

$$\phi_1(t) = \phi(-t) \quad , \quad f_1(x) = f(-x) \quad ,$$

$$\psi_1(t) = \psi(t) \quad , \quad g_1(x) = -g(x) \quad .$$

The systems noted above are not, of course, the only non-linear systems that can be solved by using the technique of Section 2. The Hardy-Poincaré-Bertrand formula can be used to linearize a number of other systems in which the non-linearity is due to the presence of a product in which an unknown function is

multiplied by an unknown Hilbert transform or an unknown Stieltjes transform. As two additional examples we cite the systems

$$\phi(x)\int_{-1}^{1}\frac{\psi(t)dt}{t-x} = f(x) + \lambda_1\phi(x)$$

$$\psi(x)\int_{-1}^{1}\frac{\phi(t)dt}{t-x} = g(x) + \lambda_2\psi(x)$$

and

$$-\phi(x)\int_{0}^{1}\frac{\phi(t)dt}{t-x} + \psi(x)\int_{0}^{1}\frac{\psi(t)dt}{t+x} = f_1(x)$$

$$\phi(x)\int_{0}^{1}\frac{\psi(t)dt}{t-x} - \psi(x)\int_{0}^{1}\frac{\phi(t)dt}{t-x} = f_2(x) \ .$$

The last system plays an important role in the theory of radiative transfer.

References

[1] Radiative Transfer by S. Chandrasekhar. Oxford University Press, (1950).

[2] Basic Methods in Transfer Problems by V. Kourganoff; I. W. Busbridge (Coll.). Oxford University Press, (1952).

[3] Fox, C., A Solution of Chandrasekhar's Integral Equation, Trans. Amer. Math. Soc. 99, (1961), 285-291.

[4] Singular Integral Equations by N. I. Muskhelishvili. P. Noordkoff N.V., Groningen, Holland (1953).

[5] Beurling, A., An Automorphism of Product Measures, Ann. of Math. (2) 72 (1960), 189-200.

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